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Annex I - "Description of Work"

GeMeThnES

Geometric Measure Theory in non-Euclidean spaces

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Principal Investigator: Luigi Ambrosio

Host Institution: Scuola Normale Superiore Pisa, Italy

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Research proposal (Part B1)

Geometric Measure Theory in non Euclidean Spaces
GeMeThnES

Principal Investigator: Luigi Ambrosio
PI's host institution for the project: Scuola Normale Superiore, Pisa, Italy
Full title of the project: Geometric Measure Theory in non Euclidean Spaces
Acronym of the project: GeMeThnES
Project duration: 60 months.

Proposal summary

Geometric Measure Theory and, in particular, the theory of currents, is one of the most basic tools in problems in Geometric Analysis, providing a parametric-free description of geometric objects which is very efficient in the study of convergence, concentration and cancellation effects, changes of topology, existence of solutions to Plateau's problem, etc. In the last years the PI and collaborators obtained ground-breaking results on the theory of currents in metric spaces and on the theory of surface measures in Carnot-Caratheodory spaces. The goal of the project is a wide range analysis of Geometric Measure Theory in spaces with a non-Euclidean structure, including infinite-dimensional spaces.

Section 1a: The Principal Investigator

Scientific Leadership Profile

1. Mumford-Shah problem, image segmentation and free discontinuity problems. Between 1988 and 1995, A. mainly studied a class of variational problems involving the minimization of volume and surface energies. Mumford and Shah proposed this in the framework of a variational approach to image segmentation, but this model has received also a lot of attention in fracture mechanics. The analysis of free discontinuity problems usually requires weak formulations. This is done in a paper by A. and De Giorgi, where a space SBV of "special" functions of bounded variation is proposed. A. built a general existence theory, with general bulk and surface energy densities.

A., Fusco and Pallara proved without apriori assumptions that any optimal set is $C^{1,\alpha}$ out of a closed singular set Σ, with $\mathcal{H}^1(\Sigma)=0$. This result holds in any dimension (in this case $\mathcal{H}^{n-1}(\Sigma)=0$) and it is still the only regularity result known to be true in any number of dimensions.

The theory developed by A. and collaborators is summarized in the OUP monograph, also a reference book on the theory of BV functions, with more than 400 citations.

2. Geometric evolution problems. In the last 20 years there has been an intensive research activity in geometric evolution problems. One of the most popular methods, introduced by Osher and Sethian, is based on the representation of the moving surface as the level set of a function solving an auxiliary PDE. A. and Soner extend this theory to flows of surfaces of *any* codimension. Another method, initiated by Brakke in 70's, is based on Geometric Measure Theory and has strong links with the Allard–Almgren theory of varifolds. In codimension greater than 1 there is no obvious way to relate the family of level sets to a varifold, and this was the main obstruction in the extension to codimension greater than 1 (i.e. to vector-valued maps) of Ilmanen's convergence proof of the reaction-diffusion equation

$$u_t = \Delta u - \frac{u(1-u^2)}{\epsilon^2} \tag{3}$$

to a flow by mean curvature in the sense of Brakke. A. and Soner introduce a more general theory of varifolds and prove a convergence result in codimension 2. In a recent paper, Bethuel, Orlandi and Smets, coupling the analysis of A-Soner with hard PDE estimates have obtained a general convergence result, still in codimension 2.

3. Analysis in metric spaces. In 1990, motivated by a problem in the theory of phase-transitions, A. introduced the concept of BV map between a domain $\Omega \subset \mathbf{R}^n$ and a general metric space (E,d) and the definition can be adapted to the Sobolev case. Some years later this was independently discovered and popularized by Reshetnyak, see also the OUP monograph by A. and Tilli. The theory of BV functions with values in metric spaces plays also a fundamental technical role in the Acta paper where A. and Kirchheim extend the Federer–Fleming theory of m-currents to any metric space, following an approach suggested by De Giorgi. The results of A.-Kirchheim have been a great surprise for the Geometric Measure Theory community, since all proofs of the Federer–Fleming theory were heavily using the Euclidean structure of the ambient space. This theory provides new results even in the Euclidean case and it is very well adapted to the Gromov–Hausdorff theory of convergence of metric spaces. Recently, in an Inventiones paper, Wenger used these tools to provide a sharp characterization of Gromov hyperbolic spaces in terms of the (large scale) isoperimetric constant. Using these tools A. and Kirchheim show existence results for Plateau's problem even in infinite-dimensional Banach spaces. A. has proved a general version of De Giorgi's rectifiability theorem in Ahlfors regular metric measure spaces for which an abstract version of the Poincaré inequality holds. This result opened the way to more detailed investigations of the rectifiability problem in Carnot groups (by Franchi, Serapioni and Serra Cassano), where more structure is available. In turn, these results are the basis for the Cheeger-Kleiner rigidity result, showing the impossibility to embed the Heisenberg groups into L^1 in a bi-Lipschitz way.

4. Optimal transport theory. The problem of optimal transportation, raised by Monge in 1781, has found in recent years an enormous attention, due to its connections with Calculus of Variations, Fluid Mechanics, Probability, Economics and other fields. The existence of optimal transport maps is a delicate problem. The case when the cost is a strictly convex function of the Euclidean distance has been studied, at various levels of generality, by many authors. The other cases are more delicate because the necessary and sufficient optimality conditions fail to give enough informations. In 1978 Sudakov claimed to have a solution for any distance cost function induced by a norm. An essential ingredient is his statement that if μ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^n$, then the conditional measures μ_C induced by the decomposition are absolutely continuous with respect to the Lebesgue measure on C. But, it turns out that when $n > 2$ the property claimed by Sudakov is not true even for the decomposition in segments: A., Kirchheim and Pratelli exhibit a remarkable example (based on an improvement of a construction suggested by Alberti, Kirchheim and Preiss) of a Borel family of pairwise disjoint open segments $\{l_i\}_{i\in I}$ and points $x_i \in l_i$ in $\mathbf{R}^3$ such that the Borel set $\{x_i :\ i \in I\}$ covers $\mathcal{L}^3$-almost all of the unit cube Q! Despite several contributions, the existence of optimal transport maps for cost=distance induced by a norm is still an open problem. A., Kirchheim and Pratelli introduced a new perturbation technique, based on an asymptotic developments by Γ-convergence. This idea has been subsequently used by many authors. Finally, the Birkhäuser monograph by A., Gigli and Savaré is devoted to a general theory of gradient flows in metric spaces and it is by now considered the standard reference on the subject, as acknowledged even in Villani's big monograph. Error estimates, convergence of the Euler implicit time discretization scheme and uniqueness of gradient flows are proved. The results of the book cover flows in spaces of probability, with a formalism which avoids the restriction to absolutely continuous measures and finite-dimensional spaces.

5. Transport equation, Cauchy problem and conservation laws. The extension of the DiPerna–Lions theory to BV vector fields, ensuring well-posedness of continuity and transport equations, has been an open problem since 1989, with partial contributions. In an Inventiones paper A. has been able to achieve this extension. Probably this achievement has been decisive in the attribution to A. of the Fermat prize in 2003. This result opens the possibility to solve some non-linear PDE's with BV data. A. and De Lellis use this method to give a general existence result for the Cauchy problem relative to the Keyfitz-Kranzer systems of conservation laws. Although the nonlinearity in KK has a very special form, this is the *first* general existence result for systems with more than 2 space variables and more than 2 equations.

Curriculum vitae

Born in Alba, Italy, on January 27, 1963.
1981. Winner of the selection of the Scuola Normale Superiore (third ex-aequo), in Mathematics.
1985. Diploma thesis (under the supervision of E.De Giorgi), on "Lower semicontinuity and relaxation problems in the Calculus of Variations"
1985-1988. Phd studies at the Scuola Normale Superiore.
1988. Winner in July of the selection for research assistant in the II University of Rome.
1992. Associate Professor in Pisa, Engineering Faculty.
1994. Full Professor of "Analisi Matematica" in Salerno, Engineering Faculty.
1995-98. Full Professor of "Analisi Matematica" in Pavia, Engineering Faculty.
1998-present. Full Professor of "Analisi Matematica" in the Scuola Normale Superiore.

Distinctions.
1991. "Bartolozzi" prize of the Italian Mathematical Union.
1996. National Prize for Mathematics and Mechanics of the italian Minister of Education.
1996. Invited speaker at the 2nd European congress of mathematicians in Budapest.
1999. "Caccioppoli" Prize of the Unione Matematica Italiana.
2002. Invited speaker at the International Congress of Mathematicians in Beijing, in the PDE session.
2005. Socio Corrispondente Accademia Nazionale dei Lincei, Roma.
2003. Awarded with the international Fermat Prize of the University of Toulouse (France).
2008. Plenary speaker at the V European Congress of Mathematics in Amsterdam.

Main Visiting positions.
1988. Visiting scientist at MIT, where I gave a course on the mathematical theory of image segmentation, by variational methods.
1997. Visiting scientist for one year at the MPI in Leipzig.
1998. Trimester "Mathematical questions in image processing", organized at the Institute Henri Poincaré by J.M.Morel, Y.Meyer e D.Mumford. I gave a course in "Geometric measure theory and applications to computer vision".
2002. ETH of Zurich, teacher of the NachDiplom course on the mathematical theory of optimal mass transportation.

Research fields.
– Calculus of Variations
– Geometric Measure Theory
– Partial Differential Equations
– Analysis in metric spaces
– Measure Theory and Probability.

Scientific direction. Starting from 1997 I have been the main investigator of 5 (2-year) national projects funded by the italian ministry of education. These project involve large groups of scientists, and my research group includes almost all italian experts in Calculus of Variations and Geometric Measure Theory (including Nicola Fusco, Giovanni Alberti, Giuseppe Buttazzo, Giuseppe Savaré). However, these projects have a very limited budget

per year (roughly 4KE per person per year) and do not allow, for instance, the activation of post-doc positions. The last 2006 project just expired and we applied for the 2008 grant. In the 10 years spent at the SNS I supervised more than 20 diploma theses and the following PhD theses have been discussed under my supervision:
1999. Francois Ebobisse. *Fine properties of functions of bounded deformation.* (now research assistant in Cape Town)
2002. Matteo Focardi. *Variational approximation of free discontinuity problems.* (now research assistant in Florence)
2002. Camillo De Lellis. *On the jacobian of weakly differentiable maps.* (now full professor in Zurich)
2002. Valentino Magnani. *Geometric Measure Theory on sub-Riemannian groups.* (now research assistant in Pisa)
2003. Aldo Pratelli. *Optimal transport maps and regularity of transport density.* (now research assistant in Pavia)
2006. Stefania Maniglia. *Well-posedness of solutions of transport equations.*
2006. Davide Vittone. *Submanifolds in Carnot groups.* (now research assistant in Padova)
2007. Alessio Figalli. *Optimal transportation and action-minimizing measures.* (now full professor at École Polytechnique)
2008. Gianluca Crippa. *The flow associated to weakly differentiable vector fields.* (now research assistant in Parma)
2008. Nicola Gigli. *Geometry of the space of measures endowed with the quadratic optimal transportation distance.* (now post-doc)

Scientific production. I published more than 100 research papers, not counting proceedings, and 3 international monographs:
Functions of bounded variation and free discontinuity problems. (with N.Fusco and D.Pallara) Oxford UP, 2000.
Selected topics on Analysis in metric Spaces. (with P.Tilli) Oxford Lecture Series in Mathematics, 2003.
Gradient flows in metric spaces and in spaces of probability measures. (with N.Gigli and G.Savaré) Birkhäuser, 2005 (second edition in 2008).

Collaborators. Not counting the many italian collaborators, I have published papers with Halil Mete Soner (Sabanci Univ., Istanbul), Sylvia Serfaty (Paris VI), Bruce Kleiner (Yale), Wilfrid Gangbo (Georgia Tech), Bernd Kirchheim (Oxford), Irene Fonseca (Carnegie Mellon University), Tristan Riviere (ETH), Francois Bouchut (ENS), John Hutchinson (ANU), Xavier Cabrè (ICREA).

10-year-Track-Record

In the last 10 years I changed a bit my research interests, publishing papers in geometric measure theory, partial differential equations, theory of optimal transportation, analysis in metric spaces. My non-italian collaborators and my former PhD students are listed in the CV. The most brilliant ones are for sure Camillo De Lellis and Alessio Figalli (now both professors abroad), but I would like to mention also Aldo Pratelli, Gianluca Crippa and Valentino Magnani.

1. Top 10 publications. The MSC database attributes to me 1964 citations by 847 authors (I have to add, also because of my very popular research books). Here I list the papers I tend to consider as more relevant (the list, including *only* the most recent papers, does not coincide with the one given in form A).

1. Semilinear Elliptic equations in $\mathbf{R}^3$ and a conjecture of De Giorgi. (with X.Cabré) Journal of the AMS, **13** (2000), 725–739. Cit: 50. The first proof of De Giorgi's conjecture in $\mathbf{R}^3$.

2. Rectifiable sets in metric and Banach spaces. (with B.Kirchheim) Mathematische Annalen, **318** (2000), 527–555. Cit: 43. A complete theory of rectifiability in metric and Banach spaces.

3. Currents in metric spaces. (with B.Kirchheim) Acta Math, **185** (2000), 1–80. Cit: 55. The extension of the Federer-Fleming theory of currents to arbitrary metric spaces.

4. Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces. Advances in Mathematics, **159** (2001), 51–67. Cit: 38. The first proof that surface measures in this context are comparable to codimension-1 Hausdorff measures and are asymptotically doubling.

5. Transport equation and Cauchy problem for BV vector fields. Invent. Math., **158** (2004), 227–260. Cit: 47. The first general well-posedness result for continuity and transport equations involving BV vector fields.

6. Existence of solutions for a class of hyperbolic systems of conservation laws. (with C. De Lellis) IMRN, **41** (2004), 2205–2220. Cit: 7. Existence of solutions, in any number of dimensions, of the Keyfitz-Kranzer system of conservation laws.

7. Well posedness for a class of hyperbolic systems of conservation laws in several space dimensions. (with C.De Lellis and F.Bouchut) Comm. PDE, **29** (2004), 1635–1651. Cit: 7. Extension of the results of **6**, now in an Eulerian perspective.

8. Optimal mass transportation in the Heisenberg group. (with S.Rigot) Journal of Functional Analysis, **208** (2004), 261–301. Cit: 11. The first extension of Brenier-McCann's theorem of existence of optimal transport maps to sub-Riemannian geometries.

9. Line energies for gradient vector fields in the plane (with C. De Lellis and C. Mantegazza), Calc. Var. & PDE, **9** (1999), 327–355. Cit: 39. A detailed analysis of a singular perturbation of the eikonal equation arising in the analysis of thin film blisters.

10. A geometric approach to monotone functions in $\mathbf{R}^n$ (with G.Alberti). Math. Z., **230** (1999), 259–316. Cit: 22. A complete analysis of the fine properties of monotone operators, including the important subclass of gradients of convex functions.

2. Monographs and contributions to volumes.

2000. Functions of bounded variation and free discontinuity problems. (with N.Fusco and D.Pallara) Oxford Mathematical monogrpahs, Oxford UP.

2004. Topics on analysis in metric spaces. (with P.Tilli) Oxford UP.
2005. Gradient flows in metric spaces and in spaces of probability measures. (with N.Gigli and G.Savaré) Birkhäuser, second edition in 2008.
2002. Optimal transport maps in Monge-Kantorovich problems. Higher ed. press, Proceedings ICM Beijing 2002.
2003. Lecture Notes on Optimal Transport Problems. Springer, LNM 1812, CIME series (contribution).
2003. Existence and stability results in the L^1 theory of optimal transportation. Springer, LNM 1813, CIME series (contribution).
2008. Transport equation and Cauchy problem for nonsmooth vector fields. Springer, LNM 1927, CIME series (contribution).

3. Presentations to conferences/schools.
2002. Invited speaker at the International Congress of Mathematicians in Beijing, in the PDE session.
2002. Plenary speaker at the joint AMS-UMI meeting in Pisa.
2006. Plenary speaker at the Hyp2006, international congress on hyperbolic problems, Lyon.
2008. Plenary speaker at the V European Congress of Mathematics in Amsterdam.

4. Organization of international conferences.
2003. Member of the scientific committee of the IV European Congress of Mathematics in Stockholm, 2004.
Main organizer of the international congress "Optimal Transport Theory and Applications", held once every two years in Pisa, starting from 2000 (last year the fifth conference took place).

5. Prizes/Awards/Memberships.
1999. "Caccioppoli" Prize of the Unione Matematica Italiana.
2005. Socio Corrispondente Accademia Nazionale dei Lincei, Roma.
2003. Awarded with the international Fermat Prize of the University of Toulouse (France), with the motivation: "for his remarkable contributions to Calculus of Variations, Geometric Measure Theory and their links with Partial Differential Equations"

6. Membership to Editorial boards.
1999. Editor of "Interfaces and Free Boundaries"
1999. Main Editor of the "Journal of the European Mathematical Society"
2000. Editor of "Archive for Rational Mechanics and Analysis"
2001. Editor of COCV "Control, Optimization and Calculus of Variations"
2005. Managing Editor of "Calculus of Variations and Partial Differential Equations"
2005. Editor of "Communications in Partial Differential Equations"
2007. Editor of M3AS "Mathematical Methods in Applied sciences"
2008. Editor of "Rendiconti del Circolo Matematico di Palermo"

Section 1b: Extended synopsis of the project proposal

I list here the main research themes. Having in mind a 5 year project, the range of the activities has to be wide (also to maximize the attractiveness of the project for post-doc and researchers), but there is a common denominator in all of them: BV functions, currents and tools from Geometric Measure Theory.

1. Geometric Measure Theory in Wiener spaces.

Many finite-dimensional concepts and results can be properly extended to infinite-dimensional spaces. In the particular case of Wiener spaces induced by Gaussian measures (roughly speaking the most isotropic spaces), the theory of Sobolev spaces and the properties of the (heat) Ornstein-Uhlenbeck semigroup are by now well understood [Bo]. On the other hand, much less appears to be known from the viewpoint of "classical" Geometric Measure Theory. For instance, the definition of surface area adopted in this context (for instance in the proof of concentration properties and in the proof of the isoperimetric property of halfspaces) typically uses Minkowski enlargment, and not directly a surface measure (see also [AM] for the case of noncritical level sets of smooth functions).
Recently the definition of BV function has been given in the context of Wiener spaces [F], and related to the OU semigroup (and recast in an integral-geometric perspective) in [AMMP]. However, even the analog of some basic finite-dimensional facts seems to be unknown. For instance, De Giorgi proved that the distributional derivative of a set of finite perimeter in $\mathbf{R}^n$ is concentrated on a set with finite $\mathcal{H}^{n-1}$ measure (the so-called reduced boundary), and in addition that out of this set the density is either 0 or 1. A very nice definition of $\mathcal{H}^{\infty-1}$ measure in Wiener spaces [FP], based on cylindrical approximations, might provide the path to the extension of De Giorgi's theorem to Wiener spaces. On the other hand, the very notion of density (and of Lebesgue point) is problematic in this context, since Besicovitch differentiation theorem fails. A challenging problem is to try to understand in which sense a set of finite perimeter is close "on small scales" to an halfspace near to boundary points. As in the finite-dimensional theory, this analysis might have an impact on the understanding of fine properties of Sobolev functions (in Euclidean spaces, the previous results imply that a $W^{1,1}$ function is approximately continuous $\mathcal{H}^{n-1}$-a.e.), while in the context of Wiener spaces approximate continuity of Sobolev functions is presently known only in the capacity sense.

2. Geometric Measure Theory in sub-Riemannian spaces.

Sub-Riemannian spaces and in particular Carnot groups appear in various areas of Mathematics, as Control Theory, Harmonic and Complex Analysis, subelliptic PDE's. A more recent line of investigation [Gr1] looks at these spaces as geometrically interesting in their own, and the understanding of the right notion of "regular surface" and surface measure induced by the Carnot-Carathéodory distance is crucial in this context. The currents of the metric theory [Aki] are modelled on Lipschitz embedding of Euclidean spaces into the metric space and are not adequate, by a typical rigidity property of these spaces. In the last few years [Ga] [FSSC1] a good understanding of this problem has been reached in the case of hypersurfaces, but only partial results are available for lower dimensional sets [FSSC2]. Even for the case of hypersurfaces many problems are still open. The main ones are maybe related to the structural properties of sets of finite perimeter, i.e. those sets E

whose distributional derivatives along directions in the horizontal layer are measures.
In this connection, the most important research directions are:
1. In Carnot groups, show that (in analogy to the Euclidean case) at almost every point, with respect to the surface measure, the blow-up of E is an halfspace. The question is settled in step 2 Carnot groups [FSSC3] and partial results are known in general Carnot groups.
2. In Carnot-Carathódory spaces, finding a description of the short-scale behaviour of sets of finite perimeter is a challenging problem, related to the fact that convergence has to be understood in the sense of measured Gromov-Hausdorff convergence. This is completely open.
3. A related problem is the understanding of the regularity theory of minimal surfaces. Even surfaces with constant horizontal normal may have in general a complex behaviour from the Euclidean viewpoint (i.e. they are not flat), and a long way is ahead in the development of a good regularity theory.

3. Isoperimetric inequalities.

Roughly speaking, an isoperimetric inequality states that for any k-cycle S there exists T bounding S (i.e. $\partial T = S$) such that

$$\text{(Is)} \qquad \mathbf{M}(T) \le C\big[\mathbf{M}(S)\big]^{(k+1)/k}.$$

Of course, the validity of (Is) and the value of the optimal C depend very much on the class of cycles and fillings one is interested to, and on the notion of mass $\mathbf{M}$ (i.e. surface area). The classical proof of the isoperimetric inequalities, for k-dimensional currents in $\mathbf{R}^n$, goes back to the work of Federer-Fleming [FF], and relies on the deformation theorem. Unfortunately, this proof provides isoperimetric constants which do depend on the dimension n. Almgren [Al1] proved afterwards that the sharp isoperimetric constant depends only on k. More recently, indirect and more flexible arguments, which are also applicable to more general ambient spaces, have been found: Gromov [Gr2] proved a concentration principle for sequences maximizing the isoperimetric ratio. This principle provides universal bounds on the isoperimetric constant in finite-dimensional Banach spaces V that depend only on k, and not on V. Ambrosio-Kirchheim [AKi] extended this to general classes of infinite-dimensional Banach spaces. Even more recently, Wenger [W1], [W2] has been able to combine these ideas with covering arguments, to provide a very efficient scheme for the proof of isoperimetric inequalities. Our goal is to improve these methods to prove new classes of isoperimetric inequalities. Two directions seem to be particularly promising and interesting:
(a) currents with coefficients in general groups G. Presently only the Euclidean case, still via deformation theorem, is known [Wh]. Recent progress [AKa] includes the additive group $G = \mathbf{Z}_p$ in general spaces, but a unified picture is still missing.
(b) surfaces in Carnot groups. Even the case $k = 1$, corresponding to closed loops S, is not fully understood, Allcock [All] provided this isoperimetric inequality when the ambient space is an Heisenberg group endowed with the Carnot-Carathéodory distance. Recent extensions of this result, by Magnani, are in [Ma].

4. Evolution problems in spaces of probability measures.

In the last few years, from many points of view it has proved very useful to look at the space $\mathcal{P}(M)$ of probability measures on M, where M is for instance a compact Riemannian manifold, as a "Riemannian" manifold in its own. This point of view became very famous in connection with Otto's seminal papers, showing that the heat equation arises as a gradient flow of the entropy functional if we endow $\mathcal{P}(M)$ with the quadratic optimal transportation distance W_2. Later on, this point of view has been extended to many more linear and nonlinear PDE's (porous medium, thin film, etc.), even when M itself is infinite-dimensional. The monograph by Ambrosio, Gigli and Savaré provides a systematic account of the theory of gradient flows in spaces of probability measures, with error estimates, existence, uniqueness and stability results. Another remarkable application of this viewpoint is in the series of papers by Lott, Villani, Sturm, where deep links between the geometry of M and the geometry of $\mathcal{P}(M)$ have been investigated: this leads to synthetic bounds from below on the Ricci tensor which do not require smoothness assumptions and display good stability properties with respect to Gromov-Hausdorff limits. In another direction, Ambrosio and Gangbo, and later on Gangbo, Kim and Pacini investigated evolution problems of Hamiltonian type in spaces of probability measures; the main advantage of this approach is the possibility to cover, with the same formalism, diffuse and singular measures. Even more recently, strong connections emerged between Mather's theory, optimal transportation and Geometric Measure Theory. On one hand, Bangert realized that Mather's minimization problem can also be formulated in terms of normal 1-currents (roughly speaking, superposition of rectifiable currents) in M. On the other hand, Bernard-Buffoni [BB] realized that these currents can also be interpreted in terms of interpolation in the space of probability measures, choosing as cost function in the optimal transportation problem the one naturally induced by the Lagrangian. These viewpoints provide in a very natural way paths in $\mathcal{P}(M)$, and it is tempting to describe intrinsically the minimality of these paths, possibly lifting the homological constraints of Mather's theory from M to $\mathcal{P}(M)$. In the model case $L(x, v) = |v|^2$, this calls for an investigation of the relationships between the integer homology of $(\mathcal{P}(M), W_2)$ and the real homology of M, and for a study of the intrinsic minimality properties of a closed, integer-multiplicity and one-dimensional metric current induced by a minimal invariant measure. In more general cases a distance in $\mathcal{P}(M)$ adapted to the Lagrangian L should be considered.

5. "Classical" Geometric Measure Theory and metric BV functions.

A central problem in Geometric Measure Theory is to understand the regularity of integer rectifiable currents in the Euclidean space, which arise as solutions of the classical Plateau's problem. In a famous monograph [Al2] (of about 1000 pages) Almgren proved that the singular set has codimension at most 2. Since holomorphic varieties are always area minimizing currents, this result is indeed optimal. Recently the result of Almgren has received more attention since higher codimension minimizing currents enter naturally in some geometric problems (see for instance the work of Taubes and Riviere-Tian). Recently De Lellis and Spadaro undertook the big project of finding easier approaches to Almgren's theory [DeS1], [DeS2]. For the moment they recover all the results on Q-valued functions minimizing the Dirichlet integral and provide a different, much shorter proof of Almgren's approximation with Q-valued functions of area-minimizing currents having small excess. In both papers, the metric theory of currents and that of metric–valued harmonic maps play a central role. Many of the simplifications of the first paper are in fact due to a new "intrinsic" approach, where the authors mostly follow the theory developed in the pioneering work of Ambrosio. Besides the obvious goal of giving a complete simpler account of Almgren's Theorem (and of its refinement for 2-d currents, due to Chang), this project opens many other interesting questions. For instance, some of the theory developed by De Lellis and Spadaro might be used to simplify and unify existing results in the theory of metric-space valued harmonic maps. In addition, the metric theory might be relevant to understand the other applications of multiple valued functions to geometric measure theory. Finally, these more refined techniques might lead to estimates independent of the codimension and therefore at least to a partial regularity theorem for minimal surfaces of finite dimension in infinite-dimensional Hilbert spaces.

6. Differential forms in singular complex spaces.
The equation $\overline{\partial} u = f$, where f is a $\overline{\partial}$-closed $(p,q)-$form on a complex manifold X, is a tool of great importance in Complex Analysis. The equation has been extensively studied and the situation is completely clear for complex manifolds. Many attempts have been made to extend the theory to complex singular spaces but some difficulties appear. On the other hand, the Lemma of Poincaré, one of the first crucial steps, is proved under very restrictive hypotheses. Henkin and Polyakov considered the case when X is a complex subspace of a ball of $\mathbf{C}^n$ taking on X the restrictions of the differential forms of the ambient. Another approach is to consider differential forms defined on the regular part $X_{\rm reg}$ of X with suitable vanishing order along the singular set of X. Having in mind a systematic and general treatment of the $\overline{\partial}$-cohomology, Mongodi studied in his master thesis *Forme differenziali e correnti metriche sugli spazi complessi* a strategy based on the notion of current dual to that of differential form. The theory of currents on metric spaces as developed by Ambrosio and Kirchheim seems to be the right general frame. Of course the metric must be chosen on X: special instances are the metric induced by closed embeddings in a Käelher spaces or the one of Kobayashi in the case of a hyperbolic space. The notion of (p,q)-current defined in the thesis induces Cauchy-Riemann operator $\overline{\partial}$. The theory of metric currents might provide a general frame which allows us to formulate for complex spaces (also of infinite dimension) the classical problems of the Calculus of Variations (characterization of holomorphic chains, boundaries of holomorphic chains...) and to attack the local and global solvability of the equation $\overline{\partial} u = f$.

References

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2. Project proposal, parts i and ii

I list here the main research themes. Having in mind a 5 year project, the range of the activities has to be wide (also to maximize the attractiveness of the project for post-doc and researchers), but there is a common denominator in all of them: BV functions, currents and tools from Geometric Measure Theory. The project has an obvious interdisciplinary character, since most research themes require an expertise in Geometric Measure Theory, Calculus of Variations, PDE's and Probability, and the team is designed in order to provide this expertise.

1. Geometric Measure Theory in Wiener spaces.

Team: L.Ambrosio, A.Figalli, V.Bogachev.

State of the art.

Many finite-dimensional concepts and results can be properly extended to infinite-dimensional spaces. We are particularly interested to a basic model in Probability and Statistical Mechanics, namely the Wiener space. To fix the ideas, let X be a separable Banach space and let γ be a nondegenerate Gaussian probability measure on X; also, let $H \subset X$ be the Cameron-Martin space, i.e. the space of vectors $h \in X$ such that γ is quasi-invariant along translations by h (i.e. the shifted measure $\gamma(h+B)$ is absolutely continuous with respect to γ). The theory of Sobolev spaces, i.e. functions weakly differentiable along directions in H, and the properties of the (heat) Ornstein-Uhlenbeck semigroup are by now well understood [Bo]. On the other hand, much less appears to be known from the viewpoint of "classical" Geometric Measure Theory. For instance, the definition of surface area adopted in this context (in the proof of concentration properties and in the proof of the isoperimetric property of halfspaces) typically uses Minkowski enlargement:

$$S(A) := \limsup_{r \downarrow 0} \frac{\gamma(A_r) - \gamma(A)}{r} \quad \text{with} \quad A_r := \{x + h : \ x \in A, \|h\|_H < r\}.$$

However, the notion of Minkowski enlargement is not directly related to a surface measure, not even in finite dimensions; smoothness assumptions on the set are typically required to make this connection (in infinite dimensions, see [AM] for the case of noncritical level sets of smooth functions).

On the other hand, in finite dimensions, De Giorgi proved in [Deg] a fundamental result: if the derivative $D\chi_E$ in the sense of distributions of a set $E \subset \mathbf{R}^n$ is a locally finite $\mathbf{R}^n$-valued measure, then there exist a Borel set $\mathcal{F}E$ (called by De Giorgi reduced boundary) and a Borel function $\nu : \mathcal{F}E \to \mathbf{S}^{n-1}$ (the so-called approximate unit normal) such that

$$D\chi_E(B) = \int_{B \cap \mathcal{F}E} \nu_E \, \mathcal{H}^{n-1} \qquad \text{for all } B \text{ Borel.}$$

Here I am using $\mathcal{H}^k$ to denote Hausdorff k-dimensional measure. This result marks the beginning of modern Geometric Measure Theory (and of the theory of BV functions as well), because it links a distributional viewpoint, very useful in the analysis of stability, weak convergence, etc., with a geometric and measure-theoretic one, very useful in the study of fine and small scale regularity properties. Also, this representation of $D\chi_E$ is similar to the one given by the classical Gauss-Green theorem, the only difference being that topological concepts have to be replaced by measure-theoretic ones, in order to define properly a boundary and a unit normal (and the abandon of topological concepts has been fundamental for the development of the subject). Later on, Federer related $\mathcal{F}E$ to the density properties of E proving that $\mathcal{F}E$ is equivalent, up to $\mathcal{H}^{n-1}$-negligible sets, to the essential boundary of E, i.e. to the set of points where the volume density of E is neither 0 nor 1.

Goals and methodology.
Recently the definition of BV function has been given in the context of Wiener spaces by Fukushima [F], in connection with the theory of stochastic processes. Fukushima provided also several equivalent definitions, based on cylindrical approximations or approximability by smooth functions, but the proof of their equivalence follows by a (somehow indirect) route through Dirichlet forms. In [AMMP] we recovered these equivalent definitions and we related them to the Ornstein-Uhlenbeck semigroup by adopting a more traditional integral-geometric perspective. In particular we obtained in Wiener spaces De Giorgi's definition of perimeter:

$$P(E) = |D\chi_E|(X) = \lim_{t\downarrow 0} \int_X |\nabla_H P_t \chi_E| \, d\gamma \quad \text{with } P_t \text{ Ornstein-Uhlenbeck semigroup.}$$

A very natural question now arises: is there an analog of De Giorgi's representation theorem of $D\chi_E$ in Wiener spaces?
We believe that the answer is yes, but many facts have to be properly understood. A very nice definition of $\mathcal{H}^{\infty-1}$ measure in Wiener spaces by Feyel and De la Pradelle [FP], based on cylindrical approximations, might provide the path to this extension. With this definition, the authors are able to obtain, among other things, an analog of Federer's co-area formula for Sobolev functions. On the other hand, if we look for additional developments of the theory, as for instance Federer's density result concerning sets of finite perimeter, or even the notion of rectifiable set, new difficulties appear. Indeed, the very notion of density (and of Lebesgue point) is problematic in this context, since Besicovitch differentiation theorem fails. In particular, Preiss provided in [Pr] examples of Wiener spaces for which Lebesgue continuity theorem (using the norm of X to define the balls) fails. A challenging problem is to try to understand in which sense a set of finite perimeter is close "on small scales" to an halfspace near to boundary points (this fact is the essential part in De Giorgi's proof [Deg] of the representation of $D\chi_E$). As in the finite-dimensional theory, this analysis might have an impact on the understanding of fine properties of Sobolev functions and infinite-dimensional integration by parts formulae in convex and nonsmooth domains. Indeed, in Euclidean spaces, the previous results imply that a $W^{1,1}$ function is approximately continuous $\mathcal{H}^{n-1}$-a.e., while in the context of Wiener spaces approximate continuity of Sobolev functions is presently known only in a different sense, the capacity sense, with a notion of capacity borrowed from the Sobolev space theory.

2. Geometric Measure Theory in sub-Riemannian spaces.

Team: L.Ambrosio, B.Kleiner, V.Magnani.

State of the art.

Sub-Riemannian spaces and in particular Carnot groups appear in various areas of Mathematics, as Control Theory, Harmonic and Complex Analysis, subelliptic PDE's. A more recent line of investigation [Gr1] looks at these spaces as geometrically interesting in their own, when endowed with the Carnot-Carathéodory distance d_{cc}. In this context, recall that $d_{cc}^2(x,y)$ is defined by taking the infimum of the action

$$\int_0^1 |\dot\gamma(t)|^2\,dt$$

among all absolutely continuous, or piecewise smooth, curves γ joining x to y and tangent to a bracket-generating distribution (in the case of Carnot groups the distribution corresponds to the first layer of the Lie algebra stratification).

The goal is to understand the proper concept of "intrinsically regular" surface in these spaces, having in mind that 1-dimensional regular surfaces should correspond precisely to the curves used in the definition of d_{cc}, and to develop a theory of currents, BV functions, sets of finite perimeter in this context. In this connection, recall that a theory of $W^{1,p}$ Sobolev spaces, developed in connection with PDE's and harmonic analysis in groups, is by now classical: it basically requires the existence of a weak derivative in L^p along directions tangent to the bracket generating distribution.

Having in mind these results, in the last few years the definition of BV function and of set of finite perimeter has been adapted to Carnot-Carathéodory spaces [Ga], [FSSC1], and by now it is pretty well understood; several facts (local and global isoperimetric inequalities, stability with respect to the Riemannian approximations of the sub-Riemannian structure, etc.) indicate that the theory provides the "right" (i.e. induced by d_{cc}) notion of (hyper)surface area. On the other hand, much less is known on the side of surfaces with an intermediate dimension (neither curves, nor hypersurfaces), with the only exception of the Heisenberg groups [FSCC2]. In this connection, one should remark that the metric theory [Aki] is basically not applicable to Carnot-Carathéodory metric spaces. Indeed, the theory provides compactness and closure theorems for the class of *rectifiable* currents. But, these currents are modelled on Lipschitz embedding of subsets of Euclidean spaces into the metric space and, by a typical rigidity property of CC spaces, it turns out that this class is too small (for instance, any Lipschitz map from a planar domain to the first Heisenberg group has $\mathcal{H}^2$-negligible image, and no nonzero 2-current of [Aki] can be supported on this image).

In the case of sets of finite perimeter in Carnot groups more detailed results are available. In this context we may fix an orthonormal basis $X_1,\ldots,X_m$ of the horizontal layer and say that E has finite perimeter if the derivatives $X_i\chi_E$ (in the sense of distributions) are Radon measures; then, the surface measure $|D\chi_E|$ is the total variation of the resulting $\mathbf{R}^m$-valued measure. A first basic result [Am1] shows that the surface measure $|D\chi_E|$ satisfies (here Q stands for the so-called homogeneous dimension)

$$0<\liminf_{r\downarrow 0}\frac{|D\chi_E|(B_r(x))}{r^{Q-1}}\le\limsup_{r\downarrow 0}\frac{|D\chi_E|(B_r(x))}{r^{Q-1}}<\infty\qquad\text{for } |D\chi_E|\text{-a.e. } x.$$

So $|D\chi_E|$ is asymptotically doubling and, even though Besicovitch differentiation theorem fails in Carnot groups, we can still differentiate with respect to $|D\chi_E|$ to make a more precise analysis, along the lines of De Giorgi's paper [Deg]. The goal is to show that, at $|D\chi_E|$-a.e. x, the blow-ups

$$\delta_{1/r}(x^{-1}E)$$

of the set E around x converge as $r \downarrow 0$ to an halfspace. The technical difficulty here comes from the fact that, after blow-up, we gain only monotonicity along one horizontal direction, invariance along the orthogonal horizontal directions and no information whatsoever on the remaining directions; on the other hand, since horizontal directions are bracket generating, there should be the possibility to transfer these informations to all directions. Indeed, it was proved in [FSCC3] that these conditions suffice to characterize halfspaces in step 2 Carnot groups, but do not suffice in higher step groups (as the Engel group). In [AKL], Ambrosio-Kleiner and Le Donne used the theory of tangent measures and a more refined analysis to show that, at $|D\chi_E|$-a.e. x, there exist $r_i \to 0^+$ such that $\delta_{1/r_i}(x^{-1}E)$ converge to an halfspace. But, full convergence is still open.

These investigations of the behaviour on small scales of sets of finite perimeter should be thought as a kind of geometric counterpart of the functional Rademacher's theorem, since in both cases a blow-up procedure is adopted and a simpler object is found (or expected) in the limit, a linear function or an hyperplane. Recently this analogy has been made more precise by Cheeger and Kleiner in [ChK]: they used the set-theoretic viewpoint as a replacement of the functional viewpoint in a typical case when the latter fails (Lipschitz maps with values in L^1, a space for which no Rademacher theorem holds). Since a typical geometric application of Rademacher theorems is the proof of rigidity results, they use the set theoretic viewpoint (and among other things the results in [Am1] and [FSCC3]) to "differentiate" any Lipschitz map from the Heisenberg group to L^1, showing that it can't be bi-Lipschitz. In this way they prove a conjecture by Lee and Naor with some relevance in theoretical computer science.

Having in mind this analogy, we can summarize the previous discussion by saying that, while the functional Rademacher theorem in Carnot group is known after Pansu's seminal paper [Pa], we still don't have a general answer on the set-theoretic Rademacher theorem.

Goals and methodology. In connection with the research themes we just described, the main goals are:

1. In Carnot groups, show that (in analogy to the Euclidean case) at almost every point, with respect to the surface measure, the blow-up of E is an halfspace. This would provide at a set-theoretic level a complete counterpart to Pansu's Rademacher [Pa] theorem in Carnot groups.

2. In Carnot-Carathódory spaces, find a description of the short-scale behaviour of sets of finite perimeter. This is a challenging problem, related to the fact that convergence has to be understood in the sense of measured Gromov-Hausdorff convergence. This problem, related to the previous one by the fact that Carnot groups can be thought as tangent to CC spaces, is completely open.

3. Understand the regularity theory of minimal surfaces. This problem is open even for surfaces with a constant horizontal normal (a calibration argument shows that these surfaces are indeed minimal). In general, the technical difficulty stems from the fact that

there is no obvious way to control the oscillation of the Euclidean norm with the oscillation of the horizontal normal. Some preliminary investigations suggest that in general Carnot groups an high regularity cannot be expected, while stronger results might be possible in the Heisenberg on in step 2 groups (where no example of non-flat surface with constant horizontal normal exists). In any case, a long way is ahead in the development of a good regularity theory.

3. Currents in metric spaces and Isoperimetric inequalities.

Team: L.Ambrosio, V.Magnani, S.Wenger.

State of the art.

Roughly speaking, an isoperimetric inequality states that for any k-cycle S there exists T bounding S (i.e. $\partial T = S$) such that

$$\text{(Is)} \qquad \mathbf{M}(T) \leq C\big[\mathbf{M}(S)\big]^{(k+1)/k}.$$

Of course, the validity of (Is) and the value of the optimal C depend very much on the class of cycles and fillings one is interested to, and on the notion of mass $\mathbf{M}$ (i.e. surface area). As a matter of fact, with the notable exception of Euclidean spaces, many different notions of mass are available in the literature.

The classical proof of the isoperimetric inequalities, for k-dimensional currents in $\mathbf{R}^n$, goes back to the work of Federer-Fleming [FF], and relies on the deformation theorem: the idea is to apply the theorem on a grid with size much larger than $[\mathbf{M}(S)]^{1/k}$. However, since the deformation theorem is proved by projecting S first on the $n-1$-skeleton of the grid, then on the $n-2$ skeleton and so on (until dimension k is reached), this proof provides isoperimetric constants which do depend on the dimension n. Almgren [Al1] proved afterwards that the sharp isoperimetric constant depends only on k, since the unique isoperimetric current is the unit k-disk. Almgren's proof is based on a very clever first variation argument in the class of area-minimizing surfaces with a volume constraint, but it is hard to adapt it to non-Euclidean spaces. More recently, indirect and more flexible arguments, which are also applicable to more general ambient spaces, have been found: Gromov [Gr2] proved by induction on k and by a cut and paste procedure a concentration principle for sequences maximizing the isoperimetric ratio. This principle provides universal bounds on the isoperimetric constant in finite-dimensional Banach spaces V that depend only on k, and not on V. Ambrosio-Kirchheim [AKi] extended this to general classes of infinite-dimensional Banach spaces for which a finite-dimensional approximation scheme exists. Even more recently, Wenger [W1], [W2] has been able to combine Gromov's cut and past procedure with covering arguments, to provide a very efficient scheme for the proof of isoperimetric inequalities, eventually including all Banach spaces.

In [AKi] the generality of the framework allows to consider limiting objects of pairs $(T_j,(E_j,d_j))$, where (E_j,d_j) are metric spaces and T_j are k-currents therein. By using Gromov-Hausdorff convergence, a limit $(T,(E,d))$ can be recovered, where (E,d) is precisely the Gromov-Hausdorff limit of (E_j,d_j). This technique was used in [Aki] to provide existence of solutions to Plateau's problem

$$\inf\Big\{\mathbf{M}(T):\ \partial T = S\Big\}$$

even when the ambient space V is not locally compact, and in particular when V is the dual of a separable Banach space: the strategy is to consider an equi-compact minimizing sequence for the problem, apply Gromov compactness theorem together with the closure theorems in [Aki] to provide an "abstract" limit T^*, living in an abstract metric space (E, d). Then, the linear structure of V allows to recover a minimizing T in V from T^*. More recently, this basic compactness scheme has been used in [W3] to provide a sharp characterization of Gromov-hyperbolic spaces in terms of a large scale isoperimetric constant: roughly speaking, if for some $\epsilon > 0$ a *quadratic* isoperimetric inequality

$$\mathbf{M}(T) \leq \frac{1-\epsilon}{4\pi}\big[\mathbf{M}(\gamma)\big]^2$$

holds for large closed loops γ, then the space is Gromov-hyperbolic and the inequality above improves to the *linear* isoperimetric inequality (obviously much stronger, on large scales).

Goals and methodology.

Our goal is to improve these methods to prove new classes of isoperimetric inequalities. Two directions seem to be particularly promising and interesting:

(a) currents with coefficients in normed groups G. In this case the classical viewpoint (adopted for rectifiable currents with real or integer coefficients in [FF] and [Aki]) of duality with differential forms is lost. However, this class of currents can be described by taking the completion, with respect to a suitable flat distance, of polyhedral chains with coefficients in G, see [Wh2]. Still using the deformation theorem, in Euclidean spaces an isoperimetric inequality is known [Wh1]. Recent progress [AKa] includes the additive group $G = \mathbf{Z}_p$ in general spaces, but a unified picture is still missing.

(b) surfaces in Carnot groups. Even the case $k = 1$, corresponding to closed loops S, is not fully understood, Allcock [All] provided this isoperimetric inequality when the ambient space is an Heisenberg group endowed with the Carnot-Carathéodory distance. Recent extensions of this result, by Magnani, are in [Ma]. In particular, in this paper new Lipschitz extension results are provided for maps f from the boundary of the unit 2-disk D^2 into a Carnot group G (under suitable assumptions on G), and this provides, via the area formula, the isoperimetric inequality. The higher dimensional cases of the Lipschitz extension theorem can be reduced, working in exponential coordinates, to a Lipschitz *differential inclusion*, for instance looking for a Lipschitz map $f : D^3 \to \mathbf{R}^6$ with given boundary data on S^2 and solving

$$df^1 \wedge df^4 + df^2 \wedge df^5 + df^3 \wedge df^6 = 0 \quad \text{a.e. in } D^3.$$

We hope to be able to attack this problem with the powerful theory of Lipschitz differential inclusions, see [Ki] and the many references therein.

4. Evolution problems in spaces of probability measures.

Team: L.Ambrosio, A.Mennucci, L.De Pascale, T.Pacini, N.Gigli, G.Savaré.

State of the art.

In the last few years, from many points of view it has proved very useful to look at the space $\mathcal{P}(M)$ of probability measures on M, where M is a compact Riemannian manifold, as a "Riemannian" manifold in its own. This point of view became very famous in connection with Otto's seminal papers, showing that the heat equation arises as a gradient flow of the entropy functional if we endow $\mathcal{P}(M)$ with the quadratic optimal transportation distance W_2. This distance, according to the Kantorovich formulation, is given by

$$W_2^2(\mu,\nu) := \min\left\{ \int_{M\times M} d_M^2(x,y)\,d\pi(x,y) : \ \pi \text{ admissible coupling from } \mu \text{ to } \nu \right\},$$

where d_M is the Riemannian distance and admissible couplings $\pi \in \mathcal{P}(M\times M)$ are defined by the property of having first and second marginal equal to μ and ν respectively.
Later on, this point of view has been extended by many authors to many more linear and nonlinear PDE's (porous medium, thin film, etc.), even when M itself is infinite-dimensional, considering different optimal transportation distances and different entropies. The monograph by Ambrosio, Gigli and Savaré [AGS] provides a systematic account of the theory of gradient flows in spaces of probability measures, with error estimates, existence, uniqueness and stability results. The viewpoint adopted in [AGS] is geometric in spirit and intrinsic: it starts from the characterization of absolutely continuous curves $\mu_t : [[0,1] \to \mathcal{P}(M)$ (a metric concept) in terms of $\mathcal{P}(M)$-values solutions μ_t to the continuity equation

$$\frac{d}{dt}\mu_t + \operatorname{div}(v_t\mu_t) = 0$$

(a differential concept), in the same spirit of the work by Benamou-Brenier, where transportation is viewed continuously in time and the optimal one arises from the minimization of a suitable action functional.
Another remarkable application of this viewpoint is in the series of papers by Lott, Villani, Sturm, where deep links between the geometry of M and the geometry of $\mathcal{P}(M)$ have been investigated: this leads to synthetic definitions of one-sides Ricci curvature bounds which do not require smoothness assumptions and display good stability properties with respect to Gromov-Hausdorff limits (exactly as in the Alexandrov theory, providing bounds on sectional curvature using geodesic triangles).
In another direction, in [AG] and [GKT] a theory of evolution problems of Hamiltonian type in spaces of probability measures starts to be developed; its main advantage is the possibility to cover, with the same formalism (borrowed from the differentiable calculus in $\mathcal{P}(M)$) absolutely continuous and singular measures. In particular these results show how the canonical symplectic structure can be transferred from $M = \mathbf{R}^{2d} \sim T\mathbf{R}^d$ to $\mathcal{P}(M)$, together with a theory of 1-differential forms.
Even more recently, strong connections emerged between Mather's theory, optimal transportation and Geometric Measure Theory. A typical formulation of Mather's problem is

$$\min\left\{ \int_{TM} L(t,x,v)\,d\mu : \ [\mu] = [h] \right\}$$

where μ runs among all closed probability measures in phase space TM (i.e. $\int d_x\phi(v)\,d\mu = 0$ for all ϕ), $L(t,x,v)$ is a (smooth) Lagrangian with superlinear growth in v, and $[h]$ is a given homology class. Bangert realized in [Ba] that Mather's minimization problem can also be formulated in terms of normal 1-currents (roughly speaking, continuous superposition of rectifiable currents) in M, and later on the result has been extended to more general Lagrangians in [DGG]. On the other hand, Bernard-Buffoni [BB] realized that these currents can also be interpreted in terms of interpolation in the space of probability measures, choosing as cost function in the optimal transportation problem the ones naturally induced by the Lagrangian:

$$c_s^t(x,y) := \min\left\{ \int_s^t L(\tau,\gamma(\tau),\dot\gamma(\tau))\,d\tau :\ \gamma(s)=x,\ \gamma(t)=y \right\}.$$

Goals and methodology. As we illustrated before, we can move from measures in TM to currents in M, and then to paths in $\mathcal{P}(M)$. It is tempting to describe intrinsically and geometrically the minimality of these paths, possibly lifting the homological constraints of Mather's theory from M to $\mathcal{P}(M)$. In the model case $L(x,v) = |v|^2$, this calls for an investigation of the relationships between the integer homology of $(\mathcal{P}(M), W_2)$ and the real homology of M, and for a study of the intrinsic minimality properties of a closed, integer-multiplicity and 1-dimensional (metric) integer rectifiable current in $\mathcal{P}(M)$ induced by a minimal invariant measure. In more general cases a distance in $\mathcal{P}(M)$ adapted to the Lagrangian L should be considered, along the lines of [BB]. On the technical side, we believe that 1-dimensional integer rectifiable currents in $\mathcal{P}(M)$ should be a discrete superposition of rectifiable curves (a class of curves already characterized in [AGS]), in analogy to results known in Euclidean spaces and in sufficiently nice spaces [W3]. On the other hand, much less is presently known on the structure of 2-dimensional integer rectifiable currents in $\mathcal{P}(M)$, and this is a necessary ingredient for the investigation of the homological properties. Also, the transfer of minimality properties from TM to currents in M and then to $\mathcal{P}(M)$ requires the nontrivial construction of suitable "lifting" operators.

5. "Classical" Geometric Measure Theory and metric BV functions.

Team: L.Ambrosio, C.De Lellis, F.Ghiraldin, C.Mantegazza.

State of the art.

A central problem in Geometric Measure Theory is to understand the regularity of integer rectifiable currents in the Euclidean spaces, which arise as solutions of the classical Plateau's problem. In a famous monograph [Al2] (of about 1000 pages) Almgren proved that the singular set has Hausdorff codimension at most 2. Since holomorphic varieties are always area minimizing currents, this result is indeed optimal. Recently the result of Almgren has received more attention since higher codimension minimizing currents enter naturally in some geometric problems (see for instance the work of Taubes [Ta] and Riviere-Tian [RT]).

In a recent series of papers, De Lellis and Spadaro are trying to find easier approaches to Almgren's theory and, at this moment, big simplifications have been obtained in the first three chapters of the book. In [DeS1] they recover all the results of Almgren on Q-valued functions minimizing the Dirichlet integral and improve some of his most important theorems. In [DeS2] they give a different and much shorter proof of Almgren's approximation with Q-valued functions of area-minimizing currents having small excess.

In this context, Q-valued functions are unordered Q-ples of points of $\mathbf{R}^N$, where $\mathbf{R}^N$ is the ambient space, endowed with the (quadratic optimal transportation) distance

$$d^2\big((x_1,\ldots,x_Q),(y_1,\ldots,y_Q)\big) = \min_{\sigma \text{ permutation}} \sum_{i=1}^{Q} |x_i - y_{\sigma(i)}|^2.$$

These objects arise very naturally in connection with regularity theory, since at branch points of the surface, where the tangent space is nearly constant, the surface leaves can be parameterized in this way.
In both papers the metric theory of currents and that of metric–valued harmonic maps play a central role. Many of the simplifications of the first paper are in fact due to a new "intrinsic" approach, where the authors build heavily upon the existing literature on metric-valued Sobolev spaces, mostly following the theory developed in the pioneering work of Ambrosio [Am] (Almgren's original approach, instead, uses a bi-Lipschitz embedding of the space $\mathcal{A}_Q(\mathbf{R}^N)$ of Q-ples in $\mathbf{R}^N$ into $\mathbf{R}^M$, with M depending on N and Q). Part of the second paper relies on the metric theory of currents [Aki] and in particular on a variant of the crucial BV estimate arising in the slicing theory of that paper.

Goals and methodology.
Besides the obvious goal of giving a complete simpler account of Almgren's Theorem (and of its refinement for 2-d currents, due to Chang [Ch]), this opens many other interesting questions. For instance, some of the theory developed in [DeS1] might be used to simplify and unify existing results in the theory of metric-space valued harmonic maps. Moreover, the second paper still uses in a crucial way a hard combinatorial lemma of Almgren's theory, which provides a powerful regularization technique for multiple valued functions. This regularization technique is the only result of Almgren's multiple valued function's theory which does not have a counterpart in the "intrinsic" theory. In addition, the metric theory might be relevant to understand the other applications of multiple valued functions to Geometric Measure Theory. Finally, these more refined techniques might lead to estimates independent of the codimension and therefore at least to a partial regularity theorem for minimal surfaces of finite dimension in infinite-dimensional Hilbert spaces.

6. Differential forms in singular complex spaces.

Team: L.Ambrosio, S.Mongodi.

State of the art.
The study of the existence problem for the equation $\overline{\partial} u = f$, where f is a $\overline{\partial}$-closed $(p,q)-$form on a complex manifold X, is a tool of great importance in Complex Analysis. More or less all crucial questions of the analytic and geometric theory of several complex variables reduce to a $\overline{\partial}$-problem, so the problem has been extensively studied and the situation is completely clear for complex manifolds. Many problems still make sense for singular complex spaces, so the extension of the theory of the sheaves of germs of differential forms on a singular complex space X is a very natural problem. Many attempts have been made to this purpose (Grauert and Kerner, Rossi, Reiffen and Vetter, Ferrari), but some difficulties appear (e.g. the presence of elements of torsion, the sheaf of germs of holomorphic p-forms is not zero if $p > \dim_{\mathbf{C}} X_{\mathrm{reg}}$). On the other hand, the Lemma of

Poincaré, one of the first crucial steps, is proved under very restrictive hypotheses. Henkin and Polyakov in [HP] considered the case when X is a complex subspace of a ball of $\mathbf{C}^n$ taking on X the restrictions of the differential forms of the ambient. Finally, in [DFV], [FG], [FYV1], [FYV2] the authors considered differential forms defined on the regular part X_{reg} of X with suitable vanishing order along the singular set of X.

Goals and methodology.

Having in mind a systematic and general treatment of the $\overline{\partial}$-cohomology, S.Mongodi in his master thesis *Forme differenziali e correnti metriche sugli spazi complessi* proposed a strategy based on the notion of current instead of that of differential form. The theory of currents in metric spaces as developed by Ambrosio and Kircheim in [Aki] seems to be the right general frame. Of course a metric must be chosen on X: special instances are the metric induced by closed embeddings in a Käelher spaces or the one of Kobayashi in the case of a hyperbolic space. After giving the notion of metric (p, q)-current, Mongodi has been able to define a Cauchy-Riemann operator $\overline{\partial}$. Then, at least in the case of completely reducible singularities (i.e. biholomorphic to a union of complex linear subspaces in some $\mathbf{C}^n$), it can be proved that the equation $\overline{\partial}u = f$ has a local solution. The global problem seems to be rather hard. Having represented metric currents on $\mathbf{C}^n$ as forms with Radon measures as coefficients, the goal is to apply some L^2-techniques (used by Hörmander in [Ho] for complex manifolds) to obtain a global solution of $\overline{\partial}u = f$ on Stein spaces. More generally the theory of metric currents provides a general frame which allows us to formulate for complex spaces (also of infinite dimension) the classical problems of the Calculus of Variations (characterization of holomorphic chains, boundaries of holomorphic chains, etc.).

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